XLI. An Analysis of the Psychometric Function for the Two-Point Limen with Respect to the Paradoxical Error

By Margaret Kincaid

We understand by the 'paradoxical error' a report of dual impression for a single-point stimulation on the skin. This 'illusion' has received attention from psychologists, and the reported experiments,[1] which have as their purpose a systematic account of the phenomenon, are evidence of its immediate impressional occurrence, apart from the effects of attentional and attitudinal variations.[2] We are proposing a mathematical analysis of the psychometric function as it is actually given by a standardized method under constant conditions; *i. e.*, we are accepting as our premise the fact of the paradoxical error and are making our problem the interpretation of this fact.

It can be seen that if the paradoxical error is strongly operative at zero separation of the aesthesiometer (a single point), the resulting psychometric function for the 'two' judgments may be hook-shaped. This fact is seen in Riecker's[3] experimental data, where the hook-curve occurs frequently. The existence of these hook-curves suggested a problem: namely, the analysis of the psychometric function into two antagonistic dispositions. Thus, in this paper, we have assumed, in the first place, a disposition *A*, which is the tendency for increase in the frequency of the judgment 'two', and which, as such, is usually thought to be the sole condition, on the side of the stimulus, of the psychometric function. We have posited, further, a disposition *B*, which represents an antagonistic tendency for the frequency of the impression 'two' to increase with smaller separations; a function that in the limiting case of zero separation might account for the paradoxical error. In brief, we have supposed that the psychometric function is a resultant of two simultaneously operative and mutually antagonistic dispositions.

Our actual experimental problem was twofold: we desired to obtain (1) data which would show an obvious tendency toward the paradoxical error, and (2) data taken under the same conditions which appeared on immediate inspection to involve no such tendency. We had, further, corresponding analytical problems: (1) to test the possibility of a mathematical resolution of the hook-curve into two hypothetical

[1] V. Henri, *Ueber die Raumwahrnehmung des Tastsinnes,* 1898, 61 ff.; V. Henri und G. Tawney, Ueber die Trugwahrnehmung zweier Punkte bei der Berührung eines Punktes der Haut, *Philos. Stud.,* 1895, 11, 394 ff.; G. A. Tawney, Ueber die Wahrnehmung zweier Punkte mittelst des Tastsinnes, mit Rücksicht auf die Frage der Uebung und die Entstehung der Vexirfehler, *Philos. Stud.,* 1896-98, 13, 163 ff.

[2] *Cf.* E. B. Titchener, *Exp. Psych.,* I., ii., 1901, 381.

[3] A. Riecker, Versuche über den Raumsinn der Kopfhaut, *Z. f. Biol.,* 1874, 10, 177 ff. There are eight hook-curves in thirty cases.

dispositions, A and B, and (2) to apply the same mathematical treatment to our 'normal' curves (*i. e.* curves without hooks), in order that the effect of analysis in the two cases might be compared.

Preliminary experiments were performed upon all available observers, in order to identify those who would be able to give the two types of data required. The observers finally selected were Miss M. Cowdrick (C), an advanced undergraduate student in psychology, and Dr. E. G. Boring (B), instructor in the department. Dr. Boring is a practised observer in cutaneous perceptions, having participated in work upon the two-point limen in two previous experiments.[4] The hook-curve which was our main objective was found for B's arm, while C's arm and B's forehead yielded 'normal' curves.

The experimental work was carried on during the Summer Session of 1917. The stimuli were applied by a Griesbach aesthesiometer with the hard rubber points used by Boring in his nerve-section experiment. The area of stimulation was controlled by a line about 8 cm. long drawn longitudinally upon the volar side of the right arm; veins, tendons and other topographical features were as far as possible avoided.[5] A similar area was employed on the forehead: a wrinkle dividing the surface approximately in half was used as the guiding line. The points were put down longitudinally with a constant pressure of 10 gr. in varying positions on or close to these lines.

The instructions given to O at the beginning of each session were as follows: "After the signal 'now', you will be given a cutaneous stimulus. Make an immediate judgment of 'one' or 'two.' The judgment 'two' is to stand for two discrete impressions; all other impressions are to be judged as 'one.' Make every judgment independently, without reference to any preceding impression. Keep as constant an attitude as possible; if for any reason your attitude changes, so that you are in doubt about a particular impression, ask to have it repeated."[6] An immediate judgment was required: if the stimulus remained upon the skin for more than 3 or 4 sec. without a report from O, the points were removed and the stimulus was applied at a different place.

TABLE I

	P=% of '2' judgments						
D=stimuli in mm.......	2	8	14	20	26	h	L
Case I. B—arm........	57	47	65	72	94	.0330	4.0316
Case II. B—forehead....	2	3	42	100	100	.1261	15.5114
Case III. C—arm.......	1	2	10	52	93	.1692	20.3833

We used the method of constant stimuli: five stimuli separated by equal steps. The size of the step was determined in preliminary experiments on B's arm. For purposes of comparison the same set

[4] E. J. Gates, The Determination of the Limens of Single and Dual Impression by the Method of Constant Stimuli, this JOURNAL, 1915, 26, 152 ff.; E. G. Boring, Cutaneous Sensation after Nerve-Division, *Quart. Jour. Exp. Physiol.*, 1916, 10, 1 ff.

[5] This area was symmetrical to the pathological area on B's left arm, Boring, *op. cit.*, 6 f. For the aesthesiometer, see p. 22.

[6] The same instructions were used with satisfactory results by L. B. Hoisington; this JOURNAL, 1917, 28, 588 ff.

of stimuli was used in the other cases. The separation of 2 mm. was as close to apposition of the aesthesiometer points as our instrument would permit. Five series of 100 stimulations each were used for the determination of a limen; one series as a rule was given on a single day.

The initial results appear in Table I. The obvious features are: (1) the inversion in case I; (2) the lack of inversions in cases II and III; (3) the high percentage of 'two' judgments with narrow separations in case I; (4) the close approach to 0% and 100% of 'two' judgments at either end of the series in cases II and III.

By inspection, cases II and III might be expected to approximate the Φ (γ)-hypothesis, and we may compute the h (measure of precision) and L (limen) for them. If we also compute these values for case I, we see that L is a very much smaller quantity; it is quite conceivable that, if the values of p were high enough, we might even have a negative limen as our result.

With these data, therefore, as an experimental background we are ready to begin our mathematical analysis. We must first, however, call attention to the fact that we might expect in these cases to get a close approximation to the $\Phi(\gamma)$-hypothesis for two reasons: (1) because it applies in other sense departments;[7] and (2) because such approximations are observed in many 'normal' cases.[8] If then in the nomenclature of our original analysis of the hook-curve we assume disposition A to be the Φ (γ), we are able to compute the B-function in accordance with the following logical argument. (1) We may expect the B-function to be least for our widest separations. (2) Assuming that the B-function may be neglected for the last two D's (separations), let us suppose that these frequencies determine a Φ (γ) which is the A-function. (Even if the B-function has not become negligible for the last two D's, we may still hope to get comparative results from this procedure; *i. e.* we may at least determine how much *more* the 'two' tendency of the B-function is for the first three D's although the absolute amount may be actually underestimated.) (3) The values of the B-function which we determine for the first three D's are the amount that the psychometric function departs from the Φ (γ) assumed for the last two D's. A word of

TABLE II

1	2	3	4	5	6	7	8	9	10	11
D	p	x	γ	γc (d=1.009)	pc	Cor-γ $(\gamma$-$\gamma c)$	Cor-x	xc (x+ cor-x)	Cor-D $\frac{6\ \text{Cor-}\gamma}{d}$	Dc D+cor-D
2	.01	—2	—1.645	—2.992	.000	1.347	1.33	—.67	11.76	13.76
8	.02	—1	—1.452	—1.983	.003	.531	.52	—.48	4.63	12.63
14	.10	0	—.906	—.974	.168	.068	.06	.06	.59	14.59
20	.52	1	.035	.035	.520	0	0	1.00	0	20
26	.93	2	1.044	1.044	.930	0	0	2.00	0	26

[7] *Cf.* F. M. Urban, The Method of Constant Stimuli and Its Generalizations, *Psychol. Rev.,* 1910, 17, 257 ff.; Ein Beitrag zur Kenntnis der psychometrischen Funktionen im Gebiete der Schallempfindungen, *Arch. f. d. ges. Psychol.,* 1910, 18, 400 ff.

[8] *Cf., e.g.,* Boring, Urban's Tables and the Method of Constant Stimuli, this JOURNAL, 1917, 28, 280 ff.; Hoisington, *op. cit.,* 594 ff.

caution at this point may not be misplaced. It must not be thought that this 'departure' could be expressed in percentages. It is *dispositions* that we are measuring; and the application of the Φ (γ)-hypothesis to the problem of the psychometric function is based on the assumption that disposition is measured in stimulus units (mm.), *i. e.* it is the amount of the change of stimulus that directly affects the disposition for a given judgment.[9]

An example of our actual method of computation is given in Table II. Starting with the actual γ's (col. 4) corresponding to our original p's (col. 2), we considered the last two γ's as fixed; and taking the difference between them as a unit, we spaced the other γ's at equal intervals (col. 5). We then found the necessary corrections for the γ's (col. 7) and expressed them in mm. (col. 10). These values, essential to our further procedure, will be found in col. 3, Table III. The "cor. D's" (Table II) then represent the amount that every point must be shifted in order to come on the Φ (γ)-curve. Thus in the hook-curve the value which causes the inversion must be

TABLE III

		1	2	3	4		5	6
				B-function				
Case		h	L (mm)	Actual necessary corrections	Least square equation of	corrective function $(y+b)\log x = k$	Σd^2	ε (%)
I	*B*-arm (uncorrected)	.0330	4.0316	15.49 7.94 4.78 0 0			.0344	.107
I	*B*-arm (corrected by *B*-function)	.0474	13.6960		b=4.32 k=7.84	21.73 4.36 2.52 1.78 1.29	.0527	.133
II	*B*-forehead (uncorrected)	.1261	15.5114	5.38 0 0 — —			.0501	.129 (.066)*
III	*C*-arm (uncorrected)	.1281	19.4108	11.76 4.63 .59 0 0			.1324	.210
III	*C*-arm (corrected by *B*-function)	.1692	20.3833		b=3.71 k=5.44	14.36 2.31 1.03 .47 .13	.0037	.035

* 3 observation-equations, omitting the two where the weight is zero.

[9] Boring, A Chart of the Psychometric Function, this JOURNAL, 1917, 28, 465 ff.

displaced by a large interval; *i. e.*, the tendency of the *B*-function for the judgment 'two' is greatest at small separations.

We have shown in Table III (col. 3) that all three of our cases exhibit a regularly decreasing *B*-function. Our procedure is of little value in case II, since there is only one point to show the corrective influence; two of the points have the weight zero, and two are operative in determining the Φ (γ)-curve. However, even this result conforms with the general conclusion. We had not anticipated that a *B*-function would be found in case III similar to the *B*-values in case I. The existence of such a function is an indication that the true status of affairs can not be determined by bare inspection. An analysis of Riecker's results showed that another of our expectations was equally untenable; hook-curves do not always depend upon a regularly decreasing *B*-function. Out of Riecker's eight hook-curves not one gives this type of function. Conversely, in case III above, a decreasing *B*-function may exist independently of a hook-curve. The treatment of a collection of data for the two-point limen taken in the Cornell Laboratory supports the assertion that the general form of the psychometric curve and the particular nature of the *B*-function exhibit no constant correlation.

We have endeavored further to determine whether the *B*-function can be adequately represented by any simple mathemetical hypothesis. For this purpose we employed straight lines, simple conics (hyperbolas, parabolas), and logarithmic curves. (It would of course be of no advantage to put a general conic through five known points, for it would fit exactly.) The curve which gave on the whole the most satisfactory results was $(y+b)\log x = k$. Using least-square equations, the unknowns b and k were determined and the hypothetical corrections were computed (col. 4. Tab. III). The latter may be directly compared with the actual required corrections given in col. 3.

We have assumed that the *A*-function is ideally the Φ (γ). The approximation of a given case to the Φ (γ)-hypothesis is measured by $\Sigma\ d^2$ [10] (col. 5) or more conveniently by ε (col. 6), since ε is independent of the number of observations and is expressed in percentages.[11] Thus, if the departures of the *A*-function from the actual data are taken account of by the *B*-function, ε would be small; the use of the actual corrections (col. 3) would give $\varepsilon = 0$. As a matter of fact, the logarithmic correction (col. 4), although it was the best simple formula we could obtain, increases ε in case I and decreases it in case III (col. 6). This result means that there probably is no simple equation for case I that adequately represents the required *B*-function.

Conclusions

(1) The occurrence of the paradoxical error may indicate the existence of two antagonistic functions (here called *A* and *B*).

(2) If we assume that the *A*-function, considered as the 'normal' function, is the Φ (γ), as we may do on extraneous grounds, it is then possible by mathematical analysis to obtain the actual residual values that constitute the *B*-function.

(3) We obtained, in our three cases, a regularly decreasing *B*-function, although analysis of cases in the literature (Riecker) and of data from the Cornell Laboratory shows this result not to be universal.

[10] *Cf.* Boring, this Journal, 1917, 28, 292.

[11] *Cf.* H. D. Williams, p. 222, above.

(Riecker's values were obtained under poor conditions; some of the Cornell results are based upon too few series to be finally indicative.)

(4) Such an analytical composition of the total psychometric function can not be determined by inspection.

(5) While we do not wish to generalize beyond our few cases, we conclude that mathematical analysis does, in our cases, indicate the operation of two antagonistic factors, and that such a solution suggests a dispositional or impressional account of the paradoxical error.

(6) In view of this conclusion we suggest that the Φ (γ)-hypothesis should not be applied to the problem of the two-point limen except in the light of a preliminary analysis.[12]

---

[12] A strong tendency toward the paradoxical error is easily demonstrable by the application of a single point. *B*, working with knowledge, reported 'two' (and 'three' and 'four'!) impressions for the single point. We were not able to discover any qualitative difference between these multiple perceptions and those obtained with wide separations.

ERRATUM

In the first part of the Checking Table facing p. 120 above, for $p = .44$, $X = O$, the value .8866 should be .8860. This correction has been made in the offprinted copies of the Table.